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Observability of discretized wave equations

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ABSTRACT: We establish several boundary observability results for finite-dimensional approximations of systems of strings and beams via space discretization. Our results allow us to recover the optimal observability theorems concerning the continuous case by a limit process.

Key Words: Observability, Fourier series, vibrating strings

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1. Introduction

Many research papers were devoted to the observability of distributed systems with continuous observability; see, e.g., [5], [7], [10], [11], [17] and their references. From a practical point of view it can be convenient to apply discrete time observation. Results in this direction were proven in [15]. Using another approach, developed in collaboration with C. Baiocchi [2], explicit and precise estimates were obtained in [8], which also contain time-discrete observability estimates for vector-valued functions and for functions of several variables.

A crucial assumption of the above theorems was a gap condition on the spectrum of the underlying operator. In order to solve various natural control problems, such as simultaneous observability and controllability of string or beam systems, this gap condition has been weakened in a work in collaboration with C. Baiocchi [2]; a discrete version of this result has been obtained recently in [9].

In this paper we give a further generalization of these results by also allowing space dicretization. If only the spatial variable is discretized, then letting the mesh size tend to zero we recover the usual continuous observation results for any

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$T > T_0$ where T_0 denotes the critical observability time. If both the spatial and time variables are discretized, then letting the two mesh sizes tend to zero we recover the usual continuous observation results for any $T > T_0$ again.

Our approach is first presented in Section 2 on the example of one (possibly loaded) string. In Sections 3 and 4 our method is extended and adapted to finite systems of strings or beams.

2. Observability of a spatially discretized wave equation

Consider a vibrating string of length ℓ with fixed endpoints and with initial data u_0 and u_1:

$$\begin{cases} u_{tt} - u_{xx} + au = 0 & \text{in } \mathbb{R} \times (0, \ell), \\ u(t, 0) = u(t, \ell) = 0 & \text{for } t \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{for } x \in (0, \ell). \end{cases} \tag{2.1}$$

Here a is a given real number.

We recall that for any given $u_0 \in H_0^1(0, \ell)$ and $u_1 \in L^2(0, \ell)$ there exists a unique weak solution satisfying

$$u \in C(\mathbb{R}; H_0^1(0, \ell)) \cap C^1(\mathbb{R}; L^2(0, \ell)) \cap C^2(\mathbb{R}; H^{-1}(0, \ell))$$

and

$$u_x(\cdot, 0), u_x(\cdot, \ell) \in L^2_{\text{loc}}(\mathbb{R}).$$

The solution is given by the series

$$u(t, x) = \sum_{k=1}^{\infty} \left(b_k e^{i\omega_k t} + b_{-k} e^{-i\omega_k t} \right) \sin(\mu_k x)$$

where

$$\mu_k = \frac{k\pi}{\ell}, \quad \omega_k = \sqrt{\mu_k^2 + a}$$

and the complex coefficients $b_{\pm k}$ depend on the initial data.

Furthermore, if I is a bounded interval of length $|I| > 2\ell$, then we have the estimate

$$\int_I |u_x(t, 0)|^2 \, dt \asymp \|u_0\|^2_{H_0^1(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)} \tag{2.2}$$

where the notation $A \asymp B$ means that $\alpha A \le B \le \beta A$ with suitable positive constants α, β, independent of the initial data.

Discretizing the system (2.1) according to the space variable we get for every positive integer N the following:

$$\begin{cases} (u_{tt}^N - \Delta_h u^N + au)(t, kh) = 0 & t \in \mathbb{R}, \quad k = 1, \dots, N-1, \\ u^N(t, 0) = u^N(t, \ell) = 0 & \text{for } t \in \mathbb{R}, \\ u^N(0, kh) = u_0(kh), & k = 1, \dots, N-1, \\ u_t^N(0, kh) = u_1(kh) & k = 1, \dots, N-1. \end{cases}$$

Here we use the notation $h = \ell/N$ and the approximation

$$\Delta_h v(x) = \frac{v(x+h) - 2v(x) + v(x-h)}{h^2}.$$

This problem also has a unique solution, given by the finite sum

$$u^N(t,x) = \sum_{k=1}^{N-1} \left(b_{N,k} e^{i\omega_{N,k}t} + b_{N,-k} e^{-i\omega_{N,k}t}\right) \sin(\mu_k x)$$

with

$$\mu_k = \frac{k\pi}{\ell}, \quad \omega_{N,k} = \sqrt{\left(\frac{\sin(\mu_k h/2)}{h/2}\right)^2 + a}$$

and suitable complex coefficients $b_{N,\pm k}$.

It follows from the finite-dimensional character of the latter system that

$$\int_I \left|\frac{u^N(t,h)}{h}\right|^2 dt \asymp \|u_0\|^2_{H_0^1(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)} \tag{2.3}$$

for *every* nondegenerated bounded interval.

A natural question is whether we may deduce (2.2) from (2.3) by letting $h \to 0$. If I is small, then the constants in the estimates blow up as $h \to 0$. The situation changes if I is suficiently long:

Proposition 2.1 *Fix two positive integers* $1 < N' < N$ *and choose*

$$|I| > \frac{2\pi}{\omega_{N,N'} - \omega_{N,N'-1}}. \tag{2.4}$$

There exist two positive constants $\alpha_{N,N'}$ *and* $\beta_{N,N'}$ *such that*

$$\begin{aligned}\alpha_{N,N'}\left(\|u_0\|^2_{H_0^1(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)}\right) &\le \int_I \left|\frac{u^N(t,h)}{h}\right|^2 dt \\ &\le \beta_{N,N'}\left(\|u_0\|^2_{H_0^1(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)}\right)\end{aligned}$$

for all functions of the form

$$u^N(t,x) = \sum_{k=1}^{N'} \left(b_k e^{i\omega_{N,k}t} + b_{-k} e^{-i\omega_{N,k}t}\right) \sin(\mu_k x)$$

with complex coefficients $b_{\pm k}$. *Furthermore, if* N' *is kept fixed and* $N \to \infty$, *then the constants* $\alpha_{N,N'}$ *and* $\beta_{N,N'}$ *can be chosen independently of* N.

Proof: It follows from the expression of $\omega_{N,k}$ that

$$\omega_{N,1} < \omega_{N,2} < \cdots < \omega_{N,N-1}$$

and

$$\omega_{N,2} - \omega_{N,1} > \omega_{N,3} - \omega_{N,2} > \cdots > \omega_{N,N-1} - \omega_{N,N-2}.$$

The first part of the proposition follows by applying Ingham's theorem.

The second part follows by observing that $\omega_{N,k} \to \omega_k$ for each fixed k if $N \to \infty$. □

We may deduce from the proposition the result for the continuous case as follows:

Theorem 2.2 *If $|I| > 2\ell$, then there exist two positive constants α and β such that*

$$\alpha \left(\|u_0\|^2_{H^1_0(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)} \right) \le \int_0^T |u_x(t,0)|^2 \, dt \le \beta \left(\|u_0\|^2_{H^1_0(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)} \right) \tag{2.5}$$

for all solutions of (2.1).

Proof: It suffices to establish the inequalities (2.5) for all finite sums of the form

$$u(t,x) = \sum_{k=1}^{N'} \left(b_k e^{i\omega_{N,k} t} + b_{-k} e^{-i\omega_{N,k} t} \right) \sin(\mu_k x)$$

with complex coefficients $b_{\pm k}$: the general case then follows by density.

For u given in this form, we apply Proposition 2.1 for every $N > N'$ satisfying (2.4). We have

$$\alpha_{N'} \left(\|u_0\|^2_{H^1_0(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)} \right) \le \int_I \left| \frac{u^N(t,h)}{h} \right|^2 \, dt \le \beta_{N'} \left(\|u_0\|^2_{H^1_0(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)} \right)$$

for every $N > N'$. Since $\omega_{N,k} \to \omega_k$ for $k = 1, \ldots, N'-1$, we conclude by letting $N \to \infty$. □

By applying in the proof of Proposition 2.1 above a discrete version of Ingham's theorem, established in [8], Theorem 1, we obtain the following result where both the time and space variables are discretized:

Proposition 2.3 *Fix two positive integers $1 < N' < N$, set $\gamma = \omega_{N,N'} - \omega_{N,N'-1}$, choose a positive number δ satisfying*

$$0 < \delta \le \pi/\gamma \quad \textit{and} \quad \omega_{N,N'} \le \frac{\pi}{\delta} + \frac{\gamma}{2}$$

and then a positive integer J satisfying $J\delta > \pi/\gamma$. Then all functions of the form

$$u^N(t,x) = \sum_{k=1}^{N'} \left(b_k e^{i\omega_{N,k}t} + b_{-k}e^{-i\omega_{N,k}t}\right)\sin(\mu_k x)$$

satisfy the estimates

$$\alpha\left(\|u_0\|^2_{H_0^1(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)}\right) \le \delta \sum_{j=-J}^{J} \Big|\frac{u^N(t'+j\delta,h)}{h}\Big|^2 \le \beta\left(\|u_0\|^2_{H_0^1(0,\ell)} + \|u_1\|^2_{L^2(0,\ell)}\right)$$

for every $t' \in \mathbb{R}$ with two positive constants α and β depending only on N, N', γ and $J\delta$.

Moreover, if N' is kept fixed, $\delta \to 0$ and $N \to \infty$, then the constants α and β can be chosen independently of δ and N.

3. Simultaneous observability of discretized strings

In this section we consider a finite number of vibrating strings with a common endpoint. Denoting their lengths by $\ell_1, \dots, \ell_M$ and using the discretization steps $h_j = \ell_j/N_j$ for $j = 1,\dots,M$, we now have the following systems:

$$\begin{cases} u_{j,tt} - u_{j,xx} + a_j u_j = 0 & \text{in } \mathbb{R}\times(0,\ell_j),\\ u_j(t,0) = u_j(t,\ell_j) = 0 & \text{for } t \in \mathbb{R},\\ u_j(0,x) = u_{j0}(x), \quad u_{j,t}(0,x) = u_{j1}(x) & \text{for } x \in (0,\ell_j),\\ j = 1,\dots,M \end{cases} \tag{3.1}$$

and

$$\begin{cases} (u^{N_j}_{j,tt} - \Delta_{h_j} u^{N_j}_j + a_j u^{N_j}_j)(t,kh_j) = 0,\\ u^{N_j}_j(t,0) = u^{N_j}_j(t,\ell_j) = 0,\\ u^{N_j}_j(0,kh_j) = u_{j0}(kh_j),\\ u^{N_j}_{j,t}(0,kh_j) = u_{j1}(kh_j)\\ \text{for} \quad t\in\mathbb{R}, \quad j = 1,\dots,M, \quad k = 1,\dots,N_j - 1. \end{cases} \tag{3.2}$$

Here $a_1,\dots, a_M$ are given real numbers.

In order to state our results we set

$$\mu_{j,k} = \frac{k\pi}{\ell_m}, \quad \omega_{j,k} = \sqrt{\mu_{j,k}^2 + a_j}, \quad \omega_{j,N_j,k} = \sqrt{\left(\frac{\sin\mu_{j,k}h_j/2}{h_j/2}\right)^2 + a_j}$$

and we introduce the Hilbert spaces $D^s(0,\ell_j)$ for each real number s and $j = 1,\dots,M$, obtained by completion of $C_c^\infty(0,\ell_j)$ with respect to the Euclidean norm

$$\Big\|\sum_{k=1}^{\infty} c_k \sin\mu_{j,k}x\Big\|_s := \left(\sum_{k=1}^{\infty} k^{2s}|c_k|^2\right)^{1/2}.$$

Note that we have in particular

$$D^0(0,\ell_j) = L^2(0,\ell_j), \quad D^1(0,\ell_j) = H_0^1(0,\ell_j), \quad D^2(0,\ell_j) = H^2(0,\ell_j) \cap H_0^1(0,\ell_j).$$

Using these notations, the problems (3.1) and (3.2) are well-posed for any initial data $u_{0m} \in D^s(0,\ell_m)$ and $u_{m1} \in D^{s-1}(0,\ell_m)$, $m = 1,\dots,M$, $s \in \mathbb{R}$, and the corresponding solutions are given by the formulae

$$u_j(t,x) = \sum_{k=1}^{\infty} \left(b_{j,k} e^{i\omega_{j,k} t} + b_{j,-k} e^{-i\omega_{j,k} t}\right) \sin(\mu_{j,k} x), \quad j = 1,\dots,M$$

and

$$u_j^{N_j}(t,x) = \sum_{k=1}^{N_j-1} \left(b_{j,N_j,k} e^{i\omega_{j,N_j,k} t} + b_{j,N_j,-k} e^{-i\omega_{j,N_j,k} t}\right) \sin(\mu_{j,k} x), \quad j = 1,\dots,M,$$

respectively, with suitable complex coefficients $b_{j,\pm k}$ and $b_{j,N_j,\pm k}$ depending on the initial data.

We recall from [2] and [7] the following result for the continuous case:

Theorem 3.1 *For almost all choices of $(\ell_1,\dots,\ell_M) \in (0,\infty)^M$, the solutions of (3.1) satisfy the estimates*

$$\sum_{m=1}^{M} \left(\|u_{m0}\|_s^2 + \|u_{m1}\|_{s-1}^2\right) \le c_{s,I} \int_I \Big|\sum_{m=1}^{M} u_{m,x}(t,0)\Big|^2 \, dt \tag{3.3}$$

on every interval I of length

$$|I| > 2(\ell_1 + \cdots + \ell_M),$$

for every $s < 2 - M$.

Moreover, if the numbers a_m are distinct, then the estimate (3.3) also holds in the limiting case $s = 2 - M$.

We are going to prove the following discretized version of this theorem:

Theorem 3.2 *Fix positive integers $1 < N_j' < N_j$, $j = 1,\dots,M$, and choose*

$$|I| > \sum_{j=1}^{M} \frac{2\pi}{\omega_{j,N_j,N_j'} - \omega_{j,N_j,N_j'-1}}.$$

Fix $s < 2 - M$ arbitrarily. There exist two positive constants α_s such that

$$\sum_{j=1}^{M} \|u_{j0}\|_s^2 + \|u_{j1}\|_{s-1}^2 \le \alpha_s \int_I \Big|\sum_{j=1}^{M} \frac{u_j^{N_j}(t,h_j)}{h_j}\Big|^2 \, dt$$

for all functions of the form

$$u_j^{N_j}(t,x) = \sum_{n=1}^{N_j'} \left(b_{j,k} e^{i\omega_{j,N_j,k}t} + b_{j,-k} e^{-i\omega_{j,N_j,k}t} \right) \sin(\mu_{j,k}x) \tag{3.4}$$

with complex coefficients $b_{j,\pm k}$. *Furthermore, if* N_j' *is kept fixed and* $N_j \to \infty$ *for all* j, *then the constants* α_s *can be chosen uniformly in* N.

If the numbers a_j *are distinct, then the conclusion also holds for* $s = 2 - M$.

Proof: It follows from the expression of $\omega_{j,N_j,k}$ that

$$\omega_{j,N_j,1} < \omega_{j,N_j,2} < \cdots < \omega_{j,N_j,N_j'},$$
$$\omega_{j,N_j,2} - \omega_{j,N_j,1} > \omega_{j,N_j,3} - \omega_{j,N_j,2} > \cdots > \omega_{j,N_j,N_j'} - \omega_{j,N_j,N_j'-1}$$

and

$$\omega_{j,N_j,k} \to \omega_{j,k} \quad \text{for each fixed } k \text{ if} \quad N_j \to \infty, \quad j = 1, \ldots, M.$$

Therefore the theorem follows by repeating the proof of Theorem 3.1 as given in [2] and [7]. □

In the case $M = 2$ we also have a doubly discretized version of the above results. Fix positive integers $1 < N_1' < N_1$, $1 < N_2' < N_2$ and set

$$\gamma := \min \left\{ \omega_{1,N_1,N_1'} - \omega_{1,N_1,N_1'-1}, \omega_{2,N_2,N_2'} - \omega_{2,N_2,N_2'-1} \right\}.$$

Furthermore, given $0 < \delta \le \frac{\pi}{\gamma}$ arbitrarily, fix an integer satisfying

$$J\delta > \frac{\pi}{\omega_{1,N_1,N_1'} - \omega_{1,N_1,N_1'-1}} + \frac{\pi}{\omega_{2,N_2,N_2'} - \omega_{2,N_2,N_2'-1}}.$$

Theorem 3.3 *For almost every choice of* (ℓ_1, ℓ_2), *all solutions of* (3.2) *of the form* (3.4) *satisfy the estimates*

$$\sum_{j=1}^{2} \|u_{j0}\|_s^2 + \|u_{j1}\|_{s-1}^2 \le \alpha_s \sum_{j=-J}^{J} \left| \frac{u_1^{N_1}(j\delta, h_1)}{h_1} + \frac{u_2^{N_2}(j\delta, h_2)}{h_2} \right|^2$$

for every negative real number s.

If the numbers a_j *are distinct, then the conclusion also holds for* $s = 0$.

Proof: The analogous result without space discretization eas established in [9]. The proof is easily adapted by considering functions of the form (3.4). □

4. Simultaneous observability of discretized beams

We consider in this section the following system:

$$\begin{cases} u_{j,tt} + u_{j,xxxx} = 0 & \text{in} \quad \mathbb{R} \times (0, \ell_j), \\ u_j(t,0) = u_j(t,\ell_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_{j,xx}(t,0) = u_{j,xx}(t,\ell_j) = 0 & \text{for} \quad t \in \mathbb{R}, \\ u_j(0,x) = u_{j0}(x) \quad \text{and} \quad u_{j,t}(0,x) = u_{j1}(x) & \text{for} \quad x \in (0,\ell_j), \\ j = 1, \ldots, M. \end{cases} \tag{4.1}$$

Using the notations of the preceding section this problem is well-posed for any initial data $u_{0m} \in D^s(0, \ell_m)$ and $u_{m1} \in D^{s-2}(0, \ell_m)$, $m = 1, \ldots, M$, $s \in \mathbb{R}$, and the corresponding solutions are given by the formulae

$$u_j(t,x) = \sum_{k=1}^{\infty} \left(b_{j,k} e^{i\omega_{j,k}^2 t} + b_{j,-k} e^{-i\omega_{j,k}^2 t} \right) \sin(\mu_{j,k} x), \quad j = 1, \ldots, M$$

and

$$u_j^{N_j}(t,x) = \sum_{k=1}^{N_j - 1} \left(b_{j,N_j,k} e^{i\omega_{j,N_j,k}^2 t} + b_{j,N_j,-k} e^{-i\omega_{j,N_j,k}^2 t} \right) \sin(\mu_{j,k} x), j = 1, \ldots, M$$

respectively, with suitable complex coefficients $b_{j,\pm k}$ and $b_{j,N_j,\pm k}$ depending on the initial data.

We recall from [2] and [7] the following result for the continuous case:

Theorem 4.1 *For almost all choices of* $(\ell_1, \ldots, \ell_N) \in (0, \infty)^M$, *the solutions of* (4.1) *satisfy the estimates*

$$\sum_{j=1}^{M} \left(\|u_{j0}\|_s^2 + \|u_{j1}\|_{s-2}^2 \right) \le \beta_s \int_I \left| \sum_{j=1}^{M} u_{j,x}(t,0) \right|^2 \, dt$$

on every nondegenerated bounded interval I *and for every* $s < 1$.

Now we consider the following discretization of (4.1):

$$\begin{cases} (u_{j,tt}^{N_j} + \Delta_{h_j}^2 u_j^{N_j})(t, kh_j) = 0, \\ u_j^{N_j}(t,0) = u_j^{N_j}(t,\ell_j) = 0, \\ u_{j,xx}^{N_j}(t,0) = u_{j,xx}^{N_j}(t,\ell_j) = 0, \\ u_j^{N_j}(0, kh_j) = u_{j0}(kh_j), \\ u_{j,t}^{N_j}(0, kh_j) = u_{j1}(kh_j), \\ \text{for} \quad t \in \mathbb{R}, \quad j = 1, \ldots, M, \quad k = 1, \ldots, N_j - 1. \end{cases} \tag{4.2}$$

Modifying the proof of Theorem 4.1 in [7] to the discretized case as we did in the preceding section, we obtain the following result:

Theorem 4.2 *Fix positive integers* $1 < N'_j < N_j$, $j = 1, \dots, M$. *For almost all choices of* $(\ell_1, \dots, \ell_N) \in (0, \infty)^M$, *the solutions of* (4.2) *of the form*

$$u_j^{N_j}(t,x) = \sum_{n=1}^{N'_j} \left(b_{j,k} e^{i\omega^2_{j,N_j,k} t} + b_{j,-k} e^{-i\omega^2_{j,N_j,k} t} \right) \sin(\mu_{j,k} x) \tag{4.3}$$

(with complex coefficients $b_{j,\pm k}$*) satisfy the estimates*

$$\sum_{j=1}^{M} \left(\|u_{j0}\|_s^2 + \|u_{j1}\|_{s-2}^2 \right) \le \beta_s \int_I \Big| \sum_{j=1}^{M} \frac{u_j^{N_j}(t,h_j)}{h_j} \Big|^2 \, dt$$

on every nondegenerated bounded interval I *and for every* $s < 1$.

Furthermore, if N'_j *is kept fixed and* $N_j \to \infty$ *for all* j, *then the constants* β_s *can be chosen independently of* N.

Theorem 4.1 may be deduced from Theorem 4.2 in the same way as Theorem 3.1 was deduced from Theorem 3.2 in the preceding section.

In the case $M = 2$ we also have a doubly discretized version of the above results. Fix positive integers $1 < N'_1 < N_1$, $1 < N'_2 < N_2$ and set

$$\gamma' := \min \left\{ \omega^2_{1,N_1,N'_1} - \omega^2_{1,N_1,N'_1-1}, \omega^2_{2,N_2,N'_2} - \omega^2_{2,N_2,N'_2-1} \right\}.$$

Furthermore, given $0 < \delta \le \frac{\pi}{\gamma'}$ arbitrarily, fix an integer J satisfying

$$J\delta > \frac{\pi}{\omega^2_{1,N_1,N'_1} - \omega^2_{1,N_1,N'_1-1}} + \frac{\pi}{\omega^2_{2,N_2,N'_2} - \omega^2_{2,N_2,N'_2-1}}.$$

Theorem 4.3 *For almost every choice of* (ℓ_1, ℓ_2), *all solutions of* (4.2) *of the form* (4.3) *satisfy the estimates*

$$\sum_{j=1}^{2} \|u_{j0}\|_s^2 + \|u_{j1}\|_{s-1}^2 \le \alpha_s \sum_{j=-J}^{J} \Big| \frac{u_1^{N_1}(j\delta, h_1)}{h_1} + \frac{u_2^{N_2}(j\delta, h_2)}{h_2} \Big|^2$$

for every negative real number s.

Proof: The analogous result without space discretization eas established in [9]. The proof is easily adapted by considering functions of the form (3.4). □

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